

# What are spin currents in Heisenberg magnets?

F. Schütz<sup>1,a</sup>, P. Kopietz<sup>1</sup>, and M. Kollar<sup>2</sup>

<sup>1</sup> Institut für Theoretische Physik, Universität Frankfurt, Robert-Mayer-Strasse 8, 60054 Frankfurt, Germany

<sup>2</sup> Theoretische Physik III, Elektronische Korrelationen und Magnetismus, Institut für Physik, Universität Augsburg, 86135 Augsburg, Germany

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**Abstract.** We discuss the proper definition of the spin current operator in Heisenberg magnets subject to inhomogeneous magnetic fields. We argue that only the component of the naive “current operator”  $J_{ij} \mathbf{S}_i \times \mathbf{S}_j$  in the plane spanned by the local order parameters  $\langle \mathbf{S}_i \rangle$  and  $\langle \mathbf{S}_j \rangle$  is related to real transport of magnetization. Within a mean field approximation or in the classical ground state the spin current therefore vanishes. Thus, finite spin currents are a direct manifestation of quantum correlations in the system.

**PACS.** 75.10.Jm Quantized spin models – 75.10.Pq Spin chain models – 75.30.Ds Spin waves – 73.23.Ra Persistent currents

## 1 Introduction

In a recent Letter [1] and a subsequent paper [2] we have calculated the persistent spin currents in mesoscopic Heisenberg rings subject to inhomogeneous magnetic fields. We have emphasized the close analogy between this phenomenon and the well known persistent charge currents in mesoscopic metal rings pierced by an Aharonov-Bohm flux. In the ensuing discussions with several colleagues we have become aware of the fact that the definition of the spin current operator in Heisenberg magnets subject to inhomogeneous magnetic fields is not obvious. In this note we shall attempt to clarify this point.

A related problem, which will not be discussed in this work, is the definition of the spin current operator in semi-conducting electronic systems with strong spin-orbit interactions. Recently, Rashba [3] pointed out that also for this case the precise meaning of the concept of a spin current is rather subtle. In particular, he emphasizes that spin currents in thermodynamic equilibrium, which arise with the standard definition of the spin-current operator used in the literature, are unphysical and should be regarded as background currents which do not correspond to real transport of magnetization. A clear understanding of this concept is essential for the highly active field of information processing using spin degrees of freedom subsumed under the name of spintronics [4].

For itinerant systems the spin is an intrinsic property of the charge carriers and is carried around with their motion. For localized spin systems considered here, transport of spin is a consequence of the time evolution of the magnetization. For special cases the transport can be ascribed

to the movement of quasi-particles as magnons or spinons and again a simple physical picture emerges [5].

In this context, it is also interesting to note that in effective low-energy models for ferromagnets involving only the spin degrees of freedom even the concept of the linear momentum is not well defined [6]. In general, the dynamical equation for the spin degrees of freedom have to be supplemented by kinetic equations for the underlying fermionic excitations.

At the heart of the ambiguities involved in defining a spin current operator both for itinerant systems with spin orbit interaction as well as for Heisenberg magnets in inhomogeneous fields is the fact that the magnetization is not strictly conserved for these systems. Still, the intuitive concept of magnetization transport should also be useful for these systems and one is therefore led to define effective current operators, as we will do in this note for the case of a Heisenberg magnet in an inhomogeneous magnetic field.

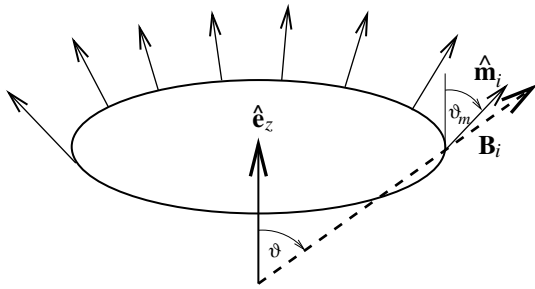
## 2 Problems with the naive definition of the spin current operator

Consider a general Heisenberg magnet with Hamiltonian

$$\hat{H} = \frac{1}{2} \sum_{i,j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - g\mu_B \sum_i \mathbf{B}_i \cdot \mathbf{S}_i, \quad (1)$$

where the sums are over all sites  $\mathbf{r}_i$  of a chain with periodic boundary conditions, the  $J_{ij}$  are general exchange couplings, and  $\mathbf{S}_i$  are spin- $S$  operators normalized such that  $\mathbf{S}_i^2 = S(S+1)$ . The last term in equation (1) is the Zeeman energy associated with an inhomogeneous magnetic field  $\mathbf{B}_i = \mathbf{B}(\mathbf{r}_i)$ . We assume that the magnetic

<sup>a</sup> e-mail: schuetz@itp.uni-frankfurt.de



**Fig. 1.** Classical spin configuration  $\hat{\mathbf{m}}_i$  of a nearest-neighbor ferromagnetic Heisenberg ring in a radial magnetic field  $\mathbf{B}_i$ .

field at each lattice site is sufficiently strong to induce permanent magnetic dipole moments  $\mathbf{m}_i = g\mu_B \langle \mathbf{S}_i \rangle$ , not necessarily parallel to  $\mathbf{B}_i$ , where  $\langle \dots \rangle$  denotes the usual thermal average. The simplest geometry is a ferromagnetic ring in a crown-shaped magnetic field, as illustrated in Figure 1. This geometry is used in the following for illustrative purposes, but our arguments are not restricted to this case. The Hamiltonian (1) implies the equation of motion

$$\hbar \frac{\partial \mathbf{S}_i}{\partial t} + \mathbf{h}_i \times \mathbf{S}_i + \sum_j J_{ij} \mathbf{S}_i \times \mathbf{S}_j = 0, \quad (2)$$

where  $\mathbf{h}_i = g\mu_B \mathbf{B}_i$ . Note that the last term in equation (2) can be written in the form  $\sum_j \mathbf{I}_{i \rightarrow j}$ , with

$$\mathbf{I}_{i \rightarrow j} = J_{ij} \mathbf{S}_i \times \mathbf{S}_j. \quad (3)$$

By analogy with the discrete lattice version of the equation of continuity for charge currents, it is tempting to identify  $\mathbf{I}_{i \rightarrow j}$  with the operator whose expectation value gives the spin current from site  $i$  to site  $j$ . In this work we shall argue that this identification is only correct for a strong *homogeneous* magnetic field, where in equilibrium the expectation values  $\langle \mathbf{S}_i \rangle$  of the spins at all sites are aligned along the same spatially constant direction of the field. Then  $\mathbf{h}_i \times \langle \mathbf{S}_i \rangle = 0$ . Using the fact that equilibrium averages are time independent,  $\frac{d}{dt} \langle \mathbf{S}_i \rangle = 0$ , we conclude from the equation of motion (2) that the lattice divergence of the spin current in the presence of a homogeneous magnetic field vanishes

$$\sum_j \langle \mathbf{I}_{i \rightarrow j} \rangle = 0. \quad (4)$$

For a one-dimensional ring with nearest neighbor hopping this implies for each site  $i$

$$\langle \mathbf{I}_{i \rightarrow i+1} \rangle + \langle \mathbf{I}_{i \rightarrow i-1} \rangle = \langle \mathbf{I}_{i \rightarrow i+1} \rangle - \langle \mathbf{I}_{i-1 \rightarrow i} \rangle = 0, \quad (5)$$

so that the same spin current  $\langle \mathbf{I} \rangle = \langle \mathbf{I}_{i \rightarrow i+1} \rangle$  flows through each link of the ring. However, the equation of motion contains only the divergence of the current, so that it does not fix the value of  $\langle \mathbf{I} \rangle$ . From the point of view of elementary vector analysis this is a consequence of the fact that both the divergence and the curl are necessary

to uniquely specify a vector field. Because the equation of motion contains only the divergence, circulating spin currents cannot be calculated using the equation of motion. In fact, even the definition of the spin current operator in a geometry permitting circulating spin currents cannot be deduced from the equation of motion. Of course, for a ring with a collinear spin configuration we know that  $\langle \mathbf{I} \rangle = 0$ , so that there are no circulating currents.

The case of a non-uniform magnetic field is more interesting. In general, the spin configuration in the ground state is then also inhomogeneous. For example, let us consider a radial magnetic field  $\mathbf{B}_i = |\mathbf{B}| \mathbf{r}_i / |\mathbf{r}_i|$  situated at constant latitude  $\vartheta_i = \vartheta$ , as shown in Figure 1. We assume that the direction  $\hat{\mathbf{m}}_i = \mathbf{m}_i / |\mathbf{m}_i|$  of the magnetic moments  $\mathbf{m}_i = g\mu_B \langle \mathbf{S}_i \rangle$  trace out a finite solid angle  $\Omega$  on the unit sphere in order-parameter space as we move once around the ring. If we consider a nearest neighbor Heisenberg ferromagnet with  $J_{ij} = -J(\delta_{i,j+1} + \delta_{i,j-1})$  then for  $|\mathbf{h}| \equiv g\mu_B |\mathbf{B}| \gtrsim JS(2\pi/N)^2$  the classical ground-state configuration  $\hat{\mathbf{m}}_i$  is radial as well, with a slightly different latitude  $\vartheta_m$  satisfying [1]

$$\sin(\vartheta_m - \vartheta) = -(JS/|\mathbf{h}|) [1 - \cos(2\pi/N)] \sin(2\vartheta_m). \quad (6)$$

The main point of this work is that in the presence of an inhomogeneous magnetic field the spin current operator is *not* simply given by equation (3). The fact that the expectation value of equation (3) cannot be a spin current is perhaps most obvious if we consider the simple case of classical spins in a star-shaped magnetic field, corresponding to  $\vartheta_m = \vartheta = \pi/2$  in Figure 1. In this case equation (3) gives for a ring with evenly spaced sites at zero temperature

$$\mathbf{I}_{i \rightarrow j} = J_{ij} \mathbf{e}_z \sin(2\pi/N), \quad (7)$$

where  $\mathbf{e}_z$  is a unit vector perpendicular to the plane of the ring. Note that at the classical level the statics and dynamics of a Heisenberg magnet are completely decoupled. Because the classical ground state does not have any intrinsic dynamics, it does not make any sense to associate a spin current with it which would correspond to moving magnetic moments. Furthermore, if the classical Heisenberg model is provided with Poisson bracket dynamics, the classical ground state yields a stationary solution, since it minimizes the energy. Clearly, such a completely stationary state cannot be used to transport magnetization. We conclude that for twisted spin configurations equation (3) is not a physically meaningful definition of the spin current operator.

### 3 Effective spin currents with correct classical limit

In order to arrive at a better definition, consider a non-equilibrium situation, i.e. start with a given density matrix at time  $t = 0$  and let the system evolve according to the unitary dynamics generated by the Hamiltonian in equation (1). The equation of motion (2) then directly

translates to a relation for the local and instantaneous order parameter

$$\partial_t \langle \mathbf{S}_i \rangle_t + \mathbf{h}_i \times \langle \mathbf{S}_i \rangle_t + \sum_j \langle \mathbf{I}_{i \rightarrow j} \rangle_t = 0. \quad (8)$$

Here,  $\langle \dots \rangle_t$  denotes an average with respect to the time dependent density matrix. It is then reasonable to demand that a transport current can lead to an accumulation of magnetization, i.e. a change in the magnitude of the local order parameter in time. For this magnitude, we obtain the equation of motion

$$\partial_t |\langle \mathbf{S}_i \rangle_t| + \sum_j \hat{\mathbf{m}}_i(t) \cdot \langle \mathbf{I}_{i \rightarrow j} \rangle_t = 0, \quad (9)$$

where  $\hat{\mathbf{m}}_i(t) = \langle \mathbf{S}_i \rangle_t / |\langle \mathbf{S}_i \rangle_t|$  is the time dependent direction of the order parameter. Note that only the longitudinal component of the naive “current operator” appears in this continuity equation without source terms. It is precisely this contribution which we have identified as the dominant one in our spin-wave calculation in [1]. The transverse components lead to a change in the direction of the local order parameter, but they are largely counteracted by the magnetic field term that acts as a source and generates a precession. If one wants to discuss the electric fields generated by the magnetization dynamics, one either has to take into account both the current  $\mathbf{I}_{i \rightarrow j}$  and the local precessional motion, or devise a way to make the cancellation explicit by including part of the “transverse current” in an effective magnetic field. We will attempt the second route here.

That this is a sensible way to proceed can be appreciated by a simple approximate calculation. In the classical ground state, the magnetization aligns parallel to the sum of the external and the exchange field. A necessary condition for the minimum of the classical energy is the invariance under small variations of the directions of the magnetization. This leads to the condition [1]

$$\mathbf{h}_i^{\text{eff}} \times \langle \mathbf{S}_i \rangle = 0, \quad \mathbf{h}_i^{\text{eff}} = \mathbf{h}_i - \sum_j J_{ij} \langle \mathbf{S}_j \rangle. \quad (10)$$

Note that the effective magnetic field contains a part of the exchange interaction, which therefore should not be included into the definition of the spin current operator to avoid double counting. Rewriting the exact equation of motion (2) in terms of the effective magnetic field  $\mathbf{h}_i^{\text{eff}}$  defined in equation (10), we obtain

$$\hbar \frac{\partial \mathbf{S}_i}{\partial t} + \mathbf{h}_i^{\text{eff}} \times \mathbf{S}_i + \sum_j \mathbf{I}_{i \rightarrow j}^{\text{eff}} = 0, \quad (11)$$

where

$$\mathbf{I}_{i \rightarrow j}^{\text{eff}} = J_{ij} \mathbf{S}_i \times [\mathbf{S}_j - \langle \mathbf{S}_j \rangle]. \quad (12)$$

Obviously,

$$\langle \mathbf{I}_{i \rightarrow j}^{\text{eff}} \rangle = J_{ij} [\langle \mathbf{S}_i \times \mathbf{S}_j \rangle - \langle \mathbf{S}_i \rangle \times \langle \mathbf{S}_j \rangle], \quad (13)$$

which vanishes identically in the classical ground state, or if the spins are treated within the mean-field approximation, where the spin correlator is factorized. Physically,

this is due to the fact that within the mean-field approximation the Heisenberg exchange interaction is replaced by an effective magnetic field, so that the different sites are uncorrelated and there are no degrees of freedom to transfer magnetization between them. In this work we discuss only localized spin models, so that charge degrees are not available to transfer magnetization between different sites.

## 4 New definition of the spin current operator

The definition of  $\mathbf{I}_{i \rightarrow j}^{\text{eff}}$  in equation (12) has the disadvantage of not being antisymmetric with respect to the exchange of the site labels, although its expectation value is obviously antisymmetric. In order to cure this problem and to generalize the concept of an effective current operator beyond the mean-field description, we propose the following definition,

$$\tilde{\mathbf{I}}_{i \rightarrow j} = \mathbf{I}_{i \rightarrow j} - \gamma_{ij} (\gamma_{ij} \cdot \mathbf{I}_{i \rightarrow j}), \quad (14)$$

with the unit vector

$$\gamma_{ij} = \frac{\mathbf{m}_i \times \mathbf{m}_j}{|\mathbf{m}_i \times \mathbf{m}_j|}. \quad (15)$$

Thus, we interpret only the projection of  $\mathbf{I}_{i \rightarrow j}$  onto the plane spanned by the two local order parameters  $\mathbf{m}_i$  and  $\mathbf{m}_j$  as a physical transport current. The contribution subtracted in equation (14) can be incorporated in an effective magnetic field. More precisely, the equilibrium expectation value of the exact equation of motion (2) can be rewritten as

$$\mathbf{h}_i^{\text{eff}} \times \langle \mathbf{S}_i \rangle + \sum_j \langle \tilde{\mathbf{I}}_{i \rightarrow j} \rangle = 0, \quad (16)$$

with the effective magnetic field now defined as

$$\mathbf{h}_i^{\text{eff}} = \mathbf{h}_i - \sum_j \frac{\langle \mathbf{S}_i \times J_{ij} \mathbf{S}_j \rangle \cdot \gamma_{ij}}{[\langle \mathbf{S}_i \rangle \times \langle \mathbf{S}_j \rangle] \cdot \gamma_{ij}} \langle \mathbf{S}_j \rangle. \quad (17)$$

This reduces to equation (10) for the classical ground state or at the mean-field level, where the correlation function in the numerator is factorized. The spin current operator defined in equation (14) is manifestly antisymmetric under the exchange of the labels, as it should be. It implicitly depends on the spin configuration via the unit vector  $\gamma_{ij}$ , so that in twisted spin configurations the spin current operator is a rather complicated functional of the exchange couplings. The fact that the current operator of an interacting many body system is a complicated functional of the interaction is well known from the theory of interacting Fermi systems [7]. In particular, when the effective interaction does not involve densities only the construction of the current operator is not straightforward [8].

For explicit calculations we use a representation of  $\tilde{\mathbf{I}}_{i \rightarrow j}$  in terms of spin operators quantized in local reference frames with the z-axes pointing along  $\hat{\mathbf{m}}_i$ , i.e. we decompose

$$\mathbf{S}_i = \sum_{\alpha=1,2,3} S_i^\alpha \mathbf{e}_i^\alpha, \quad S_i^\alpha = \mathbf{e}_i^\alpha \cdot \mathbf{S}_i, \quad (18)$$

with  $\mathbf{e}_i^3 = \hat{\mathbf{m}}_i$ . One still has a freedom in the orientation of the transverse basis  $\{\mathbf{e}_i^1, \mathbf{e}_i^2\}$ , which can elegantly be parametrized, if one uses spherical basis vectors  $\mathbf{e}_i^\pm = \mathbf{e}_i^1 \pm i\mathbf{e}_i^2$ . We can then write

$$\mathbf{e}_i^+ = e^{i\omega_{i \rightarrow j}} \tilde{\mathbf{e}}_i^+, \quad (19)$$

where  $\{\tilde{\mathbf{e}}_i^1, \tilde{\mathbf{e}}_i^2\}$  is the special transverse basis where  $\tilde{\mathbf{e}}_i^2 = \tilde{\mathbf{e}}_j^2 = \gamma_{ij}$ . With this notation we obtain the following expression for the spin current operator

$$\begin{aligned} \tilde{\mathbf{I}}_{i \rightarrow j} = & \frac{J_{ij}}{2i} \left[ S_i^- S_j^+ e^{i(\omega_{i \rightarrow j} - \omega_{j \rightarrow i})} \frac{\hat{\mathbf{m}}_i + \hat{\mathbf{m}}_j}{2} \right. \\ & - S_i^- S_j^- e^{i(\omega_{i \rightarrow j} + \omega_{j \rightarrow i})} \frac{\hat{\mathbf{m}}_i - \hat{\mathbf{m}}_j}{2} \\ & + S_i^\parallel S_j^- e^{i\omega_{i \rightarrow j}} (\gamma_{ij} \times \hat{\mathbf{m}}_i) \\ & \left. - S_i^- S_j^\parallel e^{i\omega_{j \rightarrow i}} (\gamma_{ij} \times \hat{\mathbf{m}}_j) - \text{H.c.} \right], \quad (20) \end{aligned}$$

where  $S_i^\pm = S_i^1 \pm iS_i^2 = \mathbf{e}_i^\pm \cdot \mathbf{S}_i$  are the usual ladder operators and  $S_i^\parallel = S_i^3 = \hat{\mathbf{m}}_i \cdot \mathbf{S}_i$ . The third and fourth terms in this expression couple longitudinal and transverse degrees of freedom and therefore do not contribute to leading order in a spin-wave calculation. The first and second summand are dominant for ferromagnetic and anti-ferromagnetic rings respectively and have been discussed in detail in [1] and [2]. For a magnetic field that varies smoothly as one moves through the system, the magnetic moments on neighboring lattice sites are almost collinear, so that in both cases the component of the naive ‘‘current operator’’  $\mathbf{I}_{i \rightarrow j}$  along the local order parameter is the one that really corresponds to the transport of magnetization. In [1] we had come to the same conclusion by invoking the gauge freedom in the choice of the transverse axes of quantization. Rotating the coordinate frame around  $\hat{\mathbf{m}}_i$  corresponds to the gauge transformation

$$\omega_{i \rightarrow j} \rightarrow \omega_{i \rightarrow j} + \alpha_i, \quad S_i^\pm \rightarrow S_i^\pm e^{\pm i\alpha_i}. \quad (21)$$

By this gauge freedom one is then let to identify the derivative of the Hamiltonian with respect to the gauge field  $\omega_{i \rightarrow j}$  as the relevant current operator. A comparison with equation (20) shows that this is indeed the longitudinal component of  $\mathbf{I}_{i \rightarrow j}$ . A more general gauge invariant formulation of the Heisenberg model is discussed in [9] (see also [10]). In these works, an  $O(3)$  gauge field  $\mathbf{A}_{i \rightarrow j}$  was introduced in a rather formal manner to write the Heisenberg model in a gauge invariant way and to obtain the spin stiffness tensor by means of differentiation with respect to the gauge field [11].

Note that the procedure adopted in this section is not restricted to the isotropic Heisenberg interaction of the Hamiltonian (1). For a general bilinear spin-spin interaction of the form

$$\hat{H} = \frac{1}{2} \sum_{i,j} \mathbf{S}_i \cdot \mathbf{J}_{ij} \mathbf{S}_j - \sum_i \mathbf{h}_i \cdot \mathbf{S}_i, \quad (22)$$

where  $\mathbf{J}_{ij}$  is now a  $3 \times 3$  matrix with  $\mathbf{J}_{ij}^T = \mathbf{J}_{ji}$ , the equation of motion (2) remains valid, if the expression for  $\mathbf{I}_{i \rightarrow j}$

is replaced by

$$\mathbf{I}_{i \rightarrow j} = \mathbf{S}_i \times \mathbf{J}_{ij} \mathbf{S}_j. \quad (23)$$

With this notation, equations (14–17) still hold (provided  $J_{ij}$  is replaced by the matrix  $\mathbf{J}_{ij}$ ) and part of the naive current operator  $\mathbf{I}_{i \rightarrow j}$  can again be absorbed into the definition of an effective magnetic field.

## 5 Conclusion

Let us emphasize again that our main point is rather simple: The microscopic equation of motion (2) contains only the (lattice) divergence of the spin current operator, which is not sufficient to fix its rotational part. A certain part of the operator  $\sum_j J_{ij} \mathbf{S}_i \times \mathbf{S}_j$  leads to a renormalization of the effective magnetic field and therefore should not be included into the definition of the spin current operator, see equations (10–17). The physical spin current, which corresponds to the motion of magnetic dipoles, must be defined such that a purely static twist in the ground state spin configuration of a classical Heisenberg magnet is considered to be a renormalization of the effective magnetic field, and does not contribute to the spin current. To further substantiate our proposal for an effective current operator, it would certainly be insightful to look for a more microscopic derivation by starting from a model involving charge degrees of freedom. It would also be instructive to explicitly investigate non-equilibrium situations with time dependent magnetizations.

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